Radio Spectrum Estimates Using Windowed Data and the Discrete Fourier Transform

Roger Dalke

Radio Spectrum Estimates Using Windowed Data and the Discrete Fourier Transform

Roger Dalke

U.S. DEPARTMENT OF COMMERCE

September 2010

CONTENTS

RADIO SPECTRUM ESTIMATES USING WINDOWED DATA AND THE DISCRETE FOURIER TRANSFORM

Roger Dalke*

Digital signal processing algorithms are commonly used to obtain radio spectrum estimates based on measurements. Such algorithms allow the user to apply a variety of time-domain windows and the discrete Fourier transform to RF signals and noise. The purpose of this report is to provide a description of how signal processing options such as window type, duration, and sampling rate affect power spectrum estimates. Power spectrum estimates for periodic RF signals and random processes (stationary and cyclostationary) are analyzed. The results presented can be used to select signal processing parameters and window types that minimize errors and uncertainties.

1. INTRODUCTION

Spectrum measurements are an important tool for evaluating the characteristics of radio signals and noise. Modern measurement devices employing digital signal processing technology are commonly used to obtain spectral estimates of radio signals. Implementation of this technology requires engineers to determine various measurement and signal processing parameters that meet practical requirements and at the same time do not induce deleterious artifacts. The goal of this brief report is to describe how the application of a *window*, in conjunction with the *discrete Fourier transform* (DFT), to measured data affects power spectrum estimates for periodic radio signals and radio noise. This is accomplished by comparing the theoretical power spectrum to that obtained by calculating the power spectrum after applying a window and DFT. The results presented can be used to minimize measurement errors and uncertainties. In this report, we discuss spectrum estimates for some common classes of radio signals and noise. The three classes covered are periodic signals (e.g., radar), stationary noise processes (e.g., radio receiver noise, some types of environmental noise), and cyclostationary processes (e.g., modulated communications signals, frequency hopped or gated signals).

In Section 2, we obtain expressions for spectral estimates involving periodic signals. We show that the obvious estimate of the signal line strength, i.e., the maximum value of the Fourier transform of the window (near the spectral line), contains errors due to leakage from adjacent lines and window scalloping error. Bounds on leakage errors for various window types are obtained. It is shown that, in general, the leakage error can be reduced by increasing the

---

*The author is with the Institute for Telecommunication Sciences, National Telecommunications and Information Administration, U. S. Department of Commerce, Boulder, Colorado 80305.

number of signal periods within the measurement window. Scalloping errors depend on the window type and can be significantly reduced by using a flat top window.

In Section 3 we calculate the power spectrum estimate for various random processes. First we obtain the spectrum estimate for stationary white Gaussian noise passed through an ideal anti-aliasing filter. The analysis is then extended to more general stationary and cyclostationary processes. The results presented show specifically how signal processing parameters (particularly sample rate, window type, and window duration) affect the spectrum estimate.

For the purposes of calculating power spectrum estimates, we will take the point of view that sampling rates are selected so that aliasing errors can be ignored and only the periodicity induced by sampling needs to be considered. Accordingly, analysis of spectrum estimates will be based on calculating the strength of spectral lines of the periodic extension of the windowed signal or noise. It is also assumed that the RF signals and noise are first converted to baseband, which is the starting point of our analysis. It will at times be convenient to omit limits on integrals of infinite extent. In what follows, integrals with unspecified limits are of (doubly) infinite extent.

2. SPECTRAL ESTIMATES FOR PERIODIC SIGNALS

In this section we obtain an expression for the spectral estimate based on the application of a window, $w(t)$, and DFT to a periodic RF signal. We then compare the estimate to the theoretical signal spectrum (i.e., the Fourier coefficients α_j) and describe how the choice of window and related signal processing parameters affect the estimate.

Let $\xi(t)$ be the baseband representation of a periodic radio signal with period T_s. The window duration is T and the signal is sampled with a time increment of $\Delta t = T/N$. We require that $w(t)$ be of bounded variation and that $w(t) = 0$ for $t \notin [0, T)$. It is assumed that the spectrum of the window is symmetric and is maximum at zero Hertz. We will use the following definition for the DFT of the windowed periodic signal

$$X_n = \sum_{k=0}^{N-1} w(k\Delta t)\xi(k\Delta t)e^{-i2\pi nk/N} \quad n = 0, 1, \ldots, N-1 \tag{1}$$

as the spectral estimate.

For the purposes of analysis, we need to write X_n in terms of the Fourier transforms of the window and the signal. First we define the Fourier transform of the window as

$$W(f) = \int w(t)e^{-i2\pi ft}\mathrm{d}t$$

and the Fourier transform of the periodic signal as

$$\Xi(f) = \sum_{n \in \mathbb{Z}} \alpha_n \, \delta(f - n/T_s) \tag{2}$$

where $\mathbb{Z}$ is the set of integers and

$$\alpha_n = \frac{1}{T_s} \int_0^{T_s} \xi(t)e^{-i2\pi nt/T_s}\mathrm{d}t.$$

Returning to the DFT, we can write

$$X_n = \sum_{k \in \mathbb{Z}} \phi_n(k) \tag{3}$$

where $\phi_n(k) = w(k\Delta t)\xi(k\Delta t)e^{-i2\pi kn\Delta t/T}$. The Poisson summation formula [1]

$$\sum_{k \in \mathbb{Z}} \phi_n(k) = \sum_{m \in \mathbb{Z}} \int e^{i2\pi mk}\phi_n(k)\mathrm{d}k \tag{4}$$

is used to obtain

$$X_n = \frac{N}{T} \sum_{m \in \mathbb{Z}} W * \Xi\left(\frac{n}{T} - \frac{mN}{T}\right)$$

or

$$X_n = \frac{N}{T} \begin{cases} W * \Xi \left(\frac{n}{T}\right) + \overbrace{\sum_{m \neq 0} W * \Xi \left(\frac{n}{T} - \frac{mN}{T}\right)}^{\text{alias}} & n = 0, 1, \ldots, N/2 \\[2em] W * \Xi \left(\frac{n-N}{T}\right) + \underbrace{\sum_{m \neq 1} W * \Xi \left(\frac{n}{T} - \frac{mN}{T}\right)}_{\text{alias}} & n = N/2 + 1, \ldots, N - 1 \end{cases} \tag{5}$$

where the symbol $*$ denotes convolution and N is assumed to be even. The terms marked "alias" are undesirable and result from inadequate sampling. Such errors can be minimized by sampling at such a rate that aliasing terms are negligible. Broadband signals may require the application of a filter prior to quantization to obtain, to the extent possible, a bandlimited function. In this report, we will assume that the Nyquist frequency $(N/(2T))$ is sufficiently greater than the highest significant frequency in the signal bandwidth so that aliasing errors can be ignored, i.e., for "positive" frequencies

$$X_n = \frac{N}{T} W * \Xi \left(\frac{n}{T}\right) \quad n = 0, 1, \ldots, N/2 \tag{6}$$

and for "negative" frequencies

$$X_{N-n} = \frac{N}{T} W * \Xi \left(-\frac{n}{T}\right) \quad n = 1, \ldots, N/2 - 1.$$

In what follows, we will explicitly treat only positive frequencies as the results are easily extended to negative frequencies.

We will assume that, in general, T and T_s are not commensurate and set $T = (M + \varepsilon)T_s$ where $0 \leq \varepsilon < 1$ and $M > 0$ is an integer (note that M is the number of complete signal periods in the measurement window). Using this combined with (2) and (6) yields

$$X_n = \frac{N}{T} \sum_{k \in \mathbb{Z}} \alpha_k W((n - (M + \varepsilon)k)/T). \tag{7}$$

2.1   Fourier Coefficient Estimates

Referring to (7), we see that an obvious estimate for the Fourier coefficient α_j is obtained by choosing n so that $|W((n - (M + \varepsilon)j)/T)|$ is maximum. We will therefore set $n = jM + [j\varepsilon]$, where the square brackets denote the nearest integer function, i.e.,

$$[x] = \begin{cases} \max \{n \in \mathbb{Z} | n \leq x + 1/2\} & x \geq 0 \\ \min \{n \in \mathbb{Z} | n \geq x - 1/2\} & x < 0 \end{cases},$$

in (7) and obtain the estimate

$$X_{jM+[j\varepsilon]} = \frac{N}{T} \left(\alpha_j W(([j\varepsilon] - j\varepsilon)/T) + \sum_{k \neq j} \alpha_k W((M(j - k) + [j\varepsilon] - k\varepsilon)/T) \right).$$

It is useful to express the foregoing result as

$$X_{jM+[j\varepsilon]} = \frac{N}{T}\left(\alpha_j W(\varepsilon_j/T) + \sum_{k\neq 0}\alpha_{j-k}W(((M+\varepsilon)k - \varepsilon_j)/T)\right) \tag{8}$$

where $\varepsilon_j = j\varepsilon - [j\varepsilon]$ and therefore $0 \leq |\varepsilon_j| \leq 1/2$. The first term contains exactly what we want, the Fourier coefficient α_j corresponding to frequency j/T_s. The second term is due to *leakage* from adjacent spectral lines. These undesirable contributions can be reduced by selecting a window with a steep roll off and small sidelobes. Such spurious frequency components are often referred to as *spectral leakage*. Before proceeding, some unnecessary factors can be eliminated by redefining the estimate as

$$\hat{X}_n = \frac{T}{NW(0)}X_n. \tag{9}$$

Referring to (8), we see that there are two types of errors. These will be called *leakage error* and *scalloping error* [2] and are defined below. First, the leakage error for the j th spectral line is defined as follows

$$E_j(M) = \frac{1}{W(0)}\left(\sum_{k\neq 0}\alpha_{j-k}W(((M+\varepsilon)k - \varepsilon_j)/T)\right). \tag{10}$$

Note that this error depends on the number of periods in the window (or window duration) and the frequency of the spectral line (denoted by the subscript j). The estimate for the j th spectral line can now be written as

$$\hat{X}_{jM+[j\varepsilon]} = \mathfrak{e}_j\alpha_j + E_j(M). \tag{11}$$

Scalloping error is due to the factor that scales the jth Fourier coefficient

$$\mathfrak{e}_j = \frac{W(\varepsilon_j/T)}{W(0)}. \tag{12}$$

Note that, for commonly used windows, $\mathfrak{e}_j$ is independent of M and varies with the window type and frequency of the spectral line. For typical windows, the scalloping error bounds are

$$1 \geq |\mathfrak{e}_j| \geq |W(1/(2T))/W(0)|.$$

2.1.1   Bounds on Spectral Leakage Errors

In this subsection we will obtain leakage error bounds for some common windows. We will use the standard notation $f(x) = O(g(x))$ to mean that as x tends to a limit, $f(x)/g(x)$ remains bounded, i.e., $f(x)$ is at most of the order $g(x)$. We will also use the notation $f(x) = o(g(x))$ to mean that $f(x)/g(x) \to 0$ or $f(x)$ is of smaller order than $g(x)$. The symbol $O(1)$ will be used to signify a bounded function.

Referring to (10) we easily obtain a starting point for calculating the leakage error bound

$$|E_j(M)| \leq \frac{\mathscr{C}_j}{|W(0)|} \sum_{k \neq 0} \chi(j-k) \, |W\left(((M+\varepsilon)k - \varepsilon_j)/T\right)| \tag{13}$$

where $\mathscr{C}_j = \max_{k \neq j} |\alpha_k|$ is a constant and

$$\chi(j-k) = \begin{cases} 1 & |\alpha_{j-k}| \neq 0 \\ 0 & \text{else} \end{cases}.$$

2.1.2 Uniform Window

The uniform window is defined as

$$w(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{else} \end{cases}$$

and hence, $|W(f)| = |T \operatorname{sinc}(\pi f\, T)|$. Beginning with (13), we can simplify the error bound as follows

$$
\begin{aligned}
|E_j(M)| &\leq \mathscr{C}_j \sum_{k \neq 0} \chi(j-k) \left| \frac{\sin \pi((M+\varepsilon)k - \varepsilon_j)}{\pi M k(1 + \varepsilon/M - \varepsilon_j/(Mk))} \right| \\
&\leq \frac{\mathscr{C}_j}{M\pi(1 - 1/(2M))} \sum_{k \neq 0} \frac{\chi(j-k)}{|k|}.
\end{aligned}
\tag{14}
$$

Note that $E_j \equiv 0$ when $\varepsilon = 0$ (i.e., T and T_s are commensurate).

Let $L + 1$ be the number of lines that define the nominal bandwidth of the periodic signal. The bound can then be further simplified to

$$|E_j(M)| \leq \frac{2\mathscr{C}_j H_L}{M\pi(1 - 1/(2M))} \tag{15}$$

where $H_L = \sum_{k=1}^{L} \frac{1}{k}$ are called harmonic numbers.

When L is small, H_L is easy to calculate. More generally, to describe how this bound changes with bandwidth, we can use an inequality from [3] to obtain

$$|E_j(M)| < \frac{2\mathscr{C}_j}{M\pi(1 - 1/(2M))} \left[\gamma + \log_e \left(L + \frac{1}{2} \right) + \frac{1}{24L^2} \right]$$

where $\gamma = 0.57721\ldots$ is Euler's constant. Note that this bound grows *very* slowly with the number of lines (or equivalently, bandwidth). The important point is that, for large M, the leakage error is inversely proportional to the number of signal periods in the uniform window. The asymptotic behavior of the leakage error for the uniform window can be summarized as follows:

$$|E_j(M)| \sim \begin{cases} O(1/M) & \varepsilon \neq 0 \\ 0 & \varepsilon = 0 \end{cases}.$$

These results show that the leakage error can be made as small as desired (within practical limits) by increasing M. This is not true for the scalloping error. Referring to (12), the scalloping error can be as much as 4 dB. This problem can be mitigated by using a modified version of the uniform window known as the flat top window which is discussed in the following subsection.

2.1.3 Flat Top and Related Windows

The flat top window is defined as

$$w(t) = \begin{cases} \sum_{\ell=0}^{4}(-1)^\ell a_\ell \cos 2\pi \ell t/T & 0 < t < T \\ 0 & \text{else} \end{cases}.$$

Typical values for the coefficients a_ℓ are given in the following table.

Table 1: Flat Top Window Coefficients

a_0	0.215578948
a_1	0.416631580
a_2	0.277263158
a_3	0.083578947
a_4	0.006947368

The magnitude of the Fourier transform of the flat top window is

$$|W(f)| = |T\text{sinc}(\pi \text{fT})Y(\text{f})|$$

where

$$Y(f) = \frac{1}{2}\sum_{\ell=0}^{4}(-1)^\ell a_\ell \left(\frac{1}{1 - \ell/fT} + \frac{1}{1 + \ell/fT} \right).$$

The spectrum of this window is the product of the uniform window spectrum, treated in the previous section, and $|Y(f)|$. Therefore, we need to determine how this function modifies the previous result.

Referring to (13), when $k = 1$, $fT = M + \varepsilon - \varepsilon_j$, so that for $M \leq 4$ and when $|\varepsilon - \varepsilon_j| << 1$,

$$|Y((M + \varepsilon - \varepsilon_j)/T)| = O(1/|\varepsilon - \varepsilon_j|).$$

In this case,
$$|\text{sinc}(\pi(\text{M} + \varepsilon - \varepsilon_j))\text{Y}((\text{M} + \varepsilon - \varepsilon_j)/\text{T})| = \text{O}(1).$$

Unlike the uniform window, the leakage error does not vanish when $\varepsilon = 0$. Also, in general, the magnitude of the leakage error does not decrease with M until $M > 4$. When $M >> 4$, $|Y(((M + \varepsilon)k - \varepsilon_j)/T)| = O(1)$ and the behavior is essentially the same as the uniform window, i.e.,
$$|E_j(M)| \sim O(1/M).$$

The main advantage of the flat top window is the small scalloping error. Referring to (12), the worst case scalloping error is about 0.01 dB. This is significantly less than the worst case scalloping error for the uniform window.

The Hanning and Hamming windows are of the same form with $a_2 = a_3 = a_4 = 0$. Hence, the leakage error is somewhat better for smaller values of M; however, the scalloping errors in both cases are greater than the flattop window (about 1.5 dB).

2.1.4   Ideal Gaussian Window

For this window, $|W(f)| = \sqrt{2\pi}\sigma T e^{-2(\pi\sigma fT)^2}$. We call this window ideal because the effects of time domain truncation are not included. When T is large, such effects will be small. For this window, (13) reduces to

$$|E_j(M)| \leq \mathscr{C}_j \sum_{k \neq 0} \chi(j - k)e^{-2(\pi\sigma(k(M+\varepsilon)-\varepsilon_j))^2}.$$

As before let $L + 1$ lines define the nominal bandwidth. Since $\max_j \varepsilon_j = 1/2$ and $\min \varepsilon = 0$ we have
$$|E_j(M)| \leq 2\mathscr{C}_j \sum_{k=1}^{L} e^{-2k^2(\pi\sigma(M-1/(2k)))^2}.$$

The first term of the series is $O(e^{-2(\pi\sigma(M-1/2))^2})$ and successive terms are $o(e^{-2k^2(\pi\sigma(M-1/2))^2})$. Due to the exponential character, when $\pi\sigma$ is not too small, terms involving $|k| > 1$ and $M > 1$ are negligible and it is reasonable to estimate leakage error using the adjacent lines. Hence, in general, we can say that the leakage error behaves as follows
$$|E_j(M)| \sim O(e^{-2(\pi\sigma(M-1/2))^2}).$$

The obvious advantage of the Gaussian window is that the leakage error essentially decreases exponentially with M^2.

A typical value for the noise equivalent bandwidth (see Section 3.1) of a Gaussian top window is $2.215/T$ which corresponds to $\pi\sigma = 0.4$. For this case, the maximum scalloping error is about 0.7 dB. The exponentially decreasing leakage error and small scalloping error makes the Gaussian window quite attractive.

3. SPECTRUM ESTIMATES FOR RANDOM PROCESSES

In this section, we obtain estimates for the mean power at the DFT frequencies (i.e. n/T). The estimate will be compared to the theoretical power spectrum to quantify the effects of the window and related signal processing parameters. First we will evaluate the spectral estimate for stationary Gaussian noise. Following this, we will obtain results for more general stationary and cyclostationary processes.

As in the previous section, we begin the analysis by using the Poisson summation formula ((3) and (4)). Now, however, $\xi(t)$ is assumed to be a zero-mean random process. The Fourier transform estimates obtained by applying the DFT to noise are zero-mean random variables, therefore we need to talk about statistics. For the purposes of this report, we will be content to calculate the variance.

From (3) and (4) we have

$$\text{var}\,(X_n) = \frac{N}{T} \sum_{m\in\mathbb{Z}} \text{var}\left(\int_0^T w(k)\xi(k)e^{-i2\pi k(n-mN)/T}\mathrm{d}k\right). \tag{16}$$

Before proceeding, we note that the stochastic integral exists in the sense of the quadratic mean and

$$\text{var}\left(\int_0^T w(k)\xi(k)e^{-i2\pi k(n-mN)/T}\mathrm{d}k\right) = \int_0^T\int_0^T w(k)w^*(\ell)e^{-i2\pi(k-\ell)(n-mN)/T}\gamma(k,\ell)\mathrm{d}k\mathrm{d}\ell$$

if the covariance function $\gamma(t,s) = \mathcal{E}\{\xi(t)\xi^*(s)\}$ is continuous in $[0,T]\times[0,T]$ and the Riemann integral on the right hand side exists [4]. This result extends to Riemann-Stieltjes integrals in which case $\gamma(t,s)$ needs only to be of bounded variation over the rectangle.

As in the previous section, we explicitly treat only positive frequencies and assume that effects of aliasing can be ignored (see (5)). Since $\xi(t)$ is zero mean (i.e., $\text{var}\,(X_n) = \mathcal{E}\{|X_n|^2\}$), (16) can be written in terms of Fourier transforms of the window and noise as follows

$$\mathcal{E}\left\{|\hat{X}_n|^2\right\} = \mathcal{E}\left\{\frac{1}{|W(0)|^2}\iint W(f)W^*(g)\,\Xi(n/T-f)\,\Xi^*(n/T-g)\mathrm{d}f\,\mathrm{d}g\right\}\quad n=0,\ldots,N/2-1. \tag{17}$$

where for convenience we have used $\hat{X}_n$ (see (9)). Here, we have assumed that the the Fourier transform of noise, $\Xi(f)$, exists in the sense that it can be treated as a generalized random process.

Since we will be considering broadband noise, an anti-aliasing filter needs to be used prior to quantization and the application of the DFT. We will use $r(t)$ (or $r(t,s)$ if the process is not stationary) to denote the covariance of the process prior to filtering and

$$R(f) = \int r(t)e^{-i2\pi ft}\mathrm{d}t$$

to denote the power spectrum. The notation $\xi(t)$ (and the Fourier transform $\Xi(f)$) will be used to describe the process after application of the anti-alias filter.

3.1   White Gaussian Noise

In this subsection, we consider white Gaussian noise passed through an ideal anti-alias filter $h(t)$ with Fourier transform

$$H(f) = \begin{cases} 1 & -N/(2T) \le f \le N/(2T) \\ 0 & \text{else} \end{cases}. \tag{18}$$

The noise has a constant power spectral density $R(f) = \mathfrak{N}$ and $\gamma(t) = \mathfrak{N} \int h(x)h^*(t+x)\mathrm{d}x$.

The Fourier transform of the filtered noise, $\Xi(f)$, is a generalized random process with covariance $\mathcal{E}\{\Xi(f)\Xi^*(g)\} = \mathfrak{N}|H(f)|^2\delta(f-g)$ and from (17) we obtain

$$\mathcal{E}\left\{|\hat{X}_n|^2\right\} = \frac{\mathfrak{N}}{|W(0)|^2} \int |W(f)|^2 |H(n/T - f)|^2 \mathrm{d}f. \tag{19}$$

The estimate should be proportional to $\mathfrak{N}$ for all frequencies (i.e., n/T) of interest for measurement purposes. This is accomplished by choosing a window with a reasonably steep roll off (i.e., $W(f)$ has negligible support outside of the bandpass range of the anti-alias filter in (18)), small sidelobes, and a sufficiently fast sample rate so that

$$\int |W(f)|^2 |H(n/T - f)|^2 \mathrm{d}f \approx |H(n/T)|^2 \int |W(f)|^2 \mathrm{d}f. \tag{20}$$

It is important to note that this approximation breaks down for frequencies near Nyquist due to end effects of the filter. Hence, the Nyquist frequency should be selected so that (20) is good approximation over the frequency range of interest.

We then have the desired result

$$\mathcal{E}\left\{|\hat{X}_n|^2\right\} \approx \mathfrak{N}\,B_{eq} \quad n = 0, 1, \cdots < N/2$$

where

$$B_{eq} = \frac{\int |W(f)|^2 \mathrm{d}f}{|W(0)|^2}$$

is the *equivalent noise bandwidth* of the window [2].

The total noise power in the measurement bandwidth $B_M < N/T$ is estimated by adding up the power contribution of each discrete frequency. If we let L be the number of discrete frequencies in B_M, then the power in B_M is

$$\mathscr{P} \approx \mathfrak{N}\,LB_{eq}.$$

10

At this point it is useful to introduce *normalized equivalent noise bandwidth* $B_{eq}^{(n)} = TB_{eq}$ which characterizes the window but is independent of the window duration T. The estimated power in the measurement bandwidth can now be written as

$$\mathscr{P} \approx \mathfrak{N}\frac{L}{T}B_{eq}^{(n)} \approx \mathfrak{N}B_M B_{eq}^{(n)}.$$

Hence, the estimated power in the measurement bandwidth is the actual noise power in that bandwidth scaled by the normalized equivalent noise bandwidth of the window.

Generally, the noise power density is not flat. This state of affairs is analyzed in the next subsection.

3.2 Stationary Processes

Again, the covariance is a function of a single variable and it follows that $\mathcal{E}\left\{\Xi(f)\,\Xi^*(g)\right\} = R(f)|H(f)|^2\delta(f-g)$. Here, we do not assume that $R(f)$ is constant and the noise is Gaussian. To allow for broadband noise we include an anti-alias filter as in the previous subsection (see (18)).

Equation (17) reduces to

$$\mathcal{E}\left\{|\hat{X}_n|^2\right\} = \frac{1}{|W(0)|^2}\int |W(f)|^2 R(n/T - f)|H(n/T - f)|^2\mathrm{d}f.$$

Since the actual noise power spectral density at frequency n/T is $R(n/T)$ we want

$$\int |W(f)|^2 R(n/T - f)|H(n/T - f)|^2\mathrm{d}f \approx R(n/T)|H(n/T)|^2\int |W(f)|^2\mathrm{d}f.$$

To accomplish this, we need to select a window that is narrow with a steep roll off and small sidelobes so that $R(f)$ is approximately constant over the bandwidth of the window. We also assume that the maximum frequency of interest is sufficiently below Nyquist (as in the previous subsection) so that

$$\mathcal{E}\left\{|\hat{X}_n|^2\right\} \approx R(n/T)B_{eq} \quad n = 0, 1, \cdots < N/2.$$

As before, the total power in $B_M < N/T$ is obtained by adding the contribution of each discrete frequency

$$\mathscr{P} \approx \frac{B_{eq}^{(n)}}{T}\sum_{n\in I} R(n/T)$$

where I is the interval $[-TB_M/2, TB_M/2]$.

The actual noise power for the process is $r(0) = \int R(f)\mathrm{d}f$. So, if the measurement bandwidth contains the noise power spectrum and if N and T are chosen so that

$$r(0) \approx \frac{1}{T}\sum_{n\in I} R(n/T),$$

11

the total power estimate is (approximately) proportional to the actual power as required, i.e.,

$$\mathscr{P} \approx r(0)B_{eq}^{(n)}.$$

Not surprisingly, the total power estimate contains a discrete approximation for the integral over the power spectrum. To obtain a reasonable estimate of the power, the frequency spacing $(1/T)$ should be small enough to adequately sample the power spectrum. Sampling errors can be minimized as desired by increasing T (and also N since increasing the time increment alone could lead to aliasing errors).

3.3   Cyclostationary Processes

In this subsection we extend the previous analysis to include cyclostationary processes. As before, we will allow for broadband processes and include the anti-alias filter of (18). By definition, the covariance function $r(t, s)$ is periodic in both t and s. We will assume that the Fourier series exists so that we can write

$$r(t, s) = \sum_{m \in \mathbb{Z}} r_m(s - t)e^{i2\pi mt/T_0}$$

where T_0 is the period of the covariance function and $r_m(s - t)$ are the Fourier coefficients.

Here $\xi(t)$ represents the filtered process, which is cyclostationary, with covariance $\gamma(t, s)$ and Fourier transform $\Xi(f)$. We then can write

$$
\begin{aligned}
\mathcal{E}\left\{\Xi(f)\,\Xi^*(g)\right\} &= \iint \gamma(t, s)e^{-i2\pi(ft-gs)}\mathrm{d}t\,\mathrm{d}s \\
&= \sum_{m \in \mathbb{Z}} \iiiint r_m(z)e^{i2\pi mx/T_0}h(t - x)h^*(s - z - x)e^{-i2\pi(ft-gs)}\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x
\end{aligned}
$$

and hence,

$$\mathcal{E}\left\{\Xi(f)\,\Xi^*(g)\right\} = H(f)H^*(g)\sum_{m \in \mathbb{Z}} R_m(-g)\delta(f - g - m/T_0)$$

where $R_m(f)$ is the Fourier transform of $r_m(t)$. Application of this last result to (17) gives

$$\mathcal{E}\left\{|X_n|^2\right\} = \frac{1}{|W(0)|^2}\sum_{m \in \mathbb{Z}}\int R_m(n/T - f - m/T_0)W^*(f + m/T_0)W(f)H(n/T - f)H^*(n/T - f - m/T_0)\mathrm{d}f.$$

By choosing a window with a steep roll off and small sidelobes, so that the window spectrum is negligible when $|f| \geq 1/(2T_0)$, we obtain the $m = 0$ term of the sum which is the time-average of the expected value, i.e.,

$$\left\langle \mathcal{E}\left\{|X_n|^2\right\}\right\rangle = \frac{1}{|W(0)|^2}\int R_0(n/T - f)|W(f)|^2|H(n/T - f)|^2\mathrm{d}f.$$

12

Here, the notation $\langle \cdot \rangle$ is used to emphasize the fact that this is a time average.

As in the previous subsections, the window (and window duration) should be selected so that

$$\int R_0(n/T - f)|W(f)|^2|H(n/T - f)|^2 \mathrm{d}f \approx R_0(n/T)|H(n/T)|^2 \int |W(f)|^2 \, \mathrm{d}f.$$

We then have the desired spectral estimate,

$$\left\langle \mathcal{E}\left\{|X_n|^2\right\}\right\rangle \approx R_0(n/T)B_{eq}.$$

Note that $R_0(f)$ is the *time average* power density.

The total power in $B_M < N/T$ is obtained by adding the contribution of each discrete frequency

$$\langle \mathscr{P} \rangle \approx \frac{B_{eq}^{(n)}}{T} \sum_{n \in I} R_0(n/T). \tag{21}$$

Assuming that the noise power spectrum is contained in the measurement bandwidth, we can compare this result with the actual power for the process. For this, we will use the following expression to calculate the time average of the average power of the zero-mean cyclostationary process $n(t)$:

$$\langle P \rangle = \lim_{Z \to \infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \mathcal{E}\left\{|n(t)|^2\right\} \mathrm{d}t.$$

Applying the Fourier series representation of the covariance function, we have

$$\langle P \rangle = \lim_{Z \to \infty} \sum_n r_n(0) \int_{-Z/2}^{Z/2} e^{i2\pi nt/T_0} \mathrm{d}t = \lim_{Z \to \infty} \sum_n r_n(0)\mathrm{sinc}(\pi nZ/T_0) = r_0(0)$$

(recall that $r_0(0) = \int R_0(f)\mathrm{d}f$). If the measurement bandwidth contains the power spectrum and if N and T are chosen so that

$$r_0(0) \approx \frac{1}{T} \sum_{n \in I} R_0(n/T),$$

we find that the total power estimate (21) and the average power are related as follows

$$\langle \mathscr{P} \rangle \approx r_0(0)B_{eq}^{(n)}.$$

Note that this result is similar to what we found for stationary processes. Here, the total power estimate contains a discrete approximation for the integral over the *time average* power spectrum. As before, the estimate is scaled by the normalized equivalent bandwidth of the window. Hence, a frequency spacing that adequately samples the (time average) power spectrum should be selected. Such sampling errors can be reduced as desired by increasing T (and also N since increasing the time increment alone could lead to aliasing errors).

3.4 Statistical Uncertainties

We have shown that if the window is narrow with a steep roll off, small sidelobes, and T is large enough, $\mathrm{var}\,(X_n)$ is (approximately) proportional to the power spectral density (or average power spectral density in the case of a cyclostationary process). In practice, we only obtain an estimate of $\mathrm{var}\,(X_n)$ using for example an arithmetic mean of several realizations (e.g., the Welsh method [5]). Therefore statistical uncertainties need to be considered.

The random variables $|X_n|^2$ are asymptotically (with N) independent chi-square random variables with 2 degrees of freedom and (e.g., [6])

$$\mathrm{var}\left(|\hat{X}_n|^2\right) \sim |R(n/T)B_{eq}|^2$$

for stationary processes and

$$\left\langle \mathrm{var}\left(|\hat{X}_n|^2\right)\right\rangle \sim |R_0(n/T)B_{eq}|^2$$

for cyclostationary processes.

If we denote the ith realization of X_n as $X_n^{(i)}$ and arithmetic mean as

$$C_M = \frac{1}{M}\sum_{i=1}^{M}|X_n^{(i)}|^2,$$

the C_M are asymptotically chi-square with $2M$ degrees of freedom. The estimate is biased and, to the extent that aliasing can be ignored and N is large,

$$
\begin{aligned}
\mathcal{E}\left\{C_M\right\} &\approx R(n/T)B_{eq} \\
\mathrm{var}\,(C_M) &\approx \frac{|R(n/T)B_{eq}|^2}{M}
\end{aligned}
\tag{22}
$$

for stationary processes and

$$
\begin{aligned}
\mathcal{E}\left\{C_M\right\} &\approx R_0(n/T)B_{eq} \\
\mathrm{var}\,(C_M) &\approx \frac{|R_0(n/T)B_{eq}|^2}{M}
\end{aligned}
\tag{23}
$$

for cyclostationary processes.

4. CONCLUDING REMARKS

In this report, we have described how the application of a window in conjunction with the DFT to periodic radio signals and radio noise affect power spectrum estimates. The results are used to describe how window characteristics and related signal processing parameters affect measurement errors and uncertainties.

In the case of periodic radio signals, we show that there are errors due to spectral leakage and window scalloping. These errors depend on the window type and related signal processing parameters. In particular, error bounds for various windows are presented. In all cases, the leakage error can be reduced by increasing the number of signal periods in the window (i.e., the window duration). By far, the leakage error decreases most rapidly (as a function of the window duration) for the Gaussian window. The scalloping error is independent of window duration and is smallest for the flat top window.

In the case of stationary noise, we describe how the window and related signal processing parameters affect both the estimated power spectral density and the total power in the measurement bandwidth. It is shown that the window should be selected so that the noise power spectral density is essentially constant over the bandwidth of the window. Also, the window duration should be long enough so that the noise power spectrum is adequately sampled.

Cyclostationary random processes were also considered. In this case, we examined estimates of the time average of the power spectrum. It was found that in addition to the considerations described for stationary processes, the window bandwidth should be less than (one-half) the repetition rate of the covariance function.

5. REFERENCES

[1] J. J. Benedetto and G. Zimmerman, "Sampling multipliers and the Poisson summation formula," *J. Fourier Ana. App.,* vol. 3, no. 5, pp. 505-523, 1997.

[2] F. J. Harris, "On the use of windows for harmonic analysis for the discrete Fourier transform," *Proc. of the IEEE*, vol. 66, no. 1, pp. 51-83, Jan. 1978.

[3] R.M. Young,"Euler's constant," *Math. Gaz.* 75, pp. 187-190, 1991.

[4] H. Cramer and M. R. Leadbetter, *Stationary and Related Stochastic Processes*, New York, NY: John Wiley and Sons, Inc., 1967.

[5] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*, Englewood Cliffs, NJ: Prentice-Hall, 1975, p.553.

[6] R. Dalke, "Statistical considerations for noise and interference measurements," NTIA Technical Report TR-09-458, Nov. 2008.

BIBLIOGRAPHIC DATA SHEET

1. PUBLICATION NO. TR-10-470	2. Government Accession No.	3. Recipient's Accession No.

4. TITLE AND SUBTITLE Radio Spectrum Estimates Using Windowed Data and the Discrete Fourier Transform	5. Publication Date September 2010
	6. Performing Organization Code NTIA/ITS.T

7. AUTHOR(S) Roger Dalke	9. Project/Task/Work Unit No. 3105012-300
8. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Telecommunication Sciences National Telecommunications & Information Administration U.S. Department of Commerce 325 Broadway Boulder, CO 80305	
	10. Contract/Grant No.

11. Sponsoring Organization Name and Address National Telecommunications & Information Administration Herbert C. Hoover Building 14th & Constitution Ave., NW Washington, DC 20230	12. Type of Report and Period Covered NTIA Technical Report

14. SUPPLEMENTARY NOTES

15. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)

Digital signal processing algorithms are commonly used to obtain radio spectrum estimates based on measurements. Such algorithms allow the user to apply a variety of time-domain windows and the discrete Fourier transform to RF signals and noise. The purpose of this report is to provide a description of how signal processing options such as window type, duration, and sampling rate affect power spectrum estimates. Power spectrum estimates for periodic RF signals and random processes (stationary and cyclostationary) are analyzed. The results presented can be used to select signal processing parameters and window types that minimize errors and uncertainties.

16. Key Words (Alphabetical order, separated by semicolons)
abstract; appendix; conclusion; document; figures; format; heading; introduction; policy; references; style guide; tables

17. AVAILABILITY STATEMENT ☒ UNLIMITED. ☐ FOR OFFICIAL DISTRIBUTION.	18. Security Class. (This report) Unclassified	20. Number of pages 16
	19. Security Class. (This page) Unclassified	21. Price:

NTIA FORMAL PUBLICATION SERIES

NTIA MONOGRAPH (MG)
A scholarly, professionally oriented publication dealing with state-of-the-art research or an authoritative treatment of a broad area. Expected to have long-lasting value.

NTIA SPECIAL PUBLICATION (SP)
Conference proceedings, bibliographies, selected speeches, course and instructional materials, directories, and major studies mandated by Congress.

NTIA REPORT (TR)
Important contributions to existing knowledge of less breadth than a monograph, such as results of completed projects and major activities. Subsets of this series include:

NTIA RESTRICTED REPORT (RR)
Contributions that are limited in distribution because of national security classification or Departmental constraints.

NTIA CONTRACTOR REPORT (CR)
Information generated under an NTIA contract or grant, written by the contractor, and considered an important contribution to existing knowledge.

JOINT NTIA/OTHER-AGENCY REPORT (JR)
This report receives both local NTIA and other agency review. Both agencies' logos and report series numbering appear on the cover.

NTIA SOFTWARE & DATA PRODUCTS (SD)
Software such as programs, test data, and sound/video files. This series can be used to transfer technology to U.S. industry.

NTIA HANDBOOK (HB)
Information pertaining to technical procedures, reference and data guides, and formal user's manuals that are expected to be pertinent for a long time.

NTIA TECHNICAL MEMORANDUM (TM)
Technical information typically of less breadth than an NTIA Report. The series includes data, preliminary project results, and information for a specific, limited audience.

For information about NTIA publications, contact the NTIA/ITS Technical Publications Office at 325 Broadway, Boulder, CO, 80305 Tel. (303) 497-3572 or e-mail info@its.bldrdoc.gov.

This report is for sale by the National Technical Information Service, 5285 Port Royal Road, Springfield, VA 22161,Tel. (800) 553-6847.